Errors of Cauchy.—The errors in Cauchy's work on group theory may be divided into two classes. The first and smaller class includes a number of serious errors in logic. The second class includes a large number of minor errors, some typographical, others arising through careless statement. All of the serious errors found have been noted in this paper with the exception of one to which attention has already been called.[1] He states an erroneous theorem on imprimitive groups in the following form.

If G_1 is a subgroup composed of all the substitutions that omit a given letter in a simply transitive group then G is imprimitive unless all the transitive constituents of G_1 are of the same degree.

Conclusion.—In conclusion we may say that the foundation period in the history of group theory includes the time from Lagrange to Cauchy inclusive; that at the beginning of this period group theory was a means to an end and not an end in itself. Lagrange and Ruffini thought of substitution groups only in so far as they led to practical results in the theory of equations. Galois, while broadening and deepening the application to the theory of equations may be considered as taking the initial step toward abstract group theory. In Cauchy while a group is still spoken of as the substitutions that leave a given function invariant, and the order of a group is still thought of as the number of equal values which the function can assume when the variables are permuted in every possible way, nevertheless quite as often a group is a system of conjugate substitutions and its relation to any function is entirely ignored.

In Cauchy's work it is to be noted that use is made of all the concepts originated by the earlier writers in substitution theory except those of Galois, a number of which would have proved powerful instruments in his hands, notably the idea of the invariant subgroup of which he makes no explicit use.

Because of the important theorems Cauchy proved, because of the break which he made in separating the theory of substitutions from the theory of equations and because of the importance that he attached to the theory itself, he deserves the credit as the founder of group theory.

HISTORY OF THE EXPONENETIAL AND LOGARITHMIC CONCEPTS.

By FLORIAN CAJORI, Colorado College.

V. Generalizations and Refinements Effected During the Nineteenth Century.

GRAPHIC REPRESENTATION.

The time of Wessel and Argand has been chosen as the beginning of a new epoch in our history, not because Wessel and Argand made any noteworthy advance in exponential and logarithmic theory, but because the graphic repre-

[1] See G. A. Miller in *Bibliotheca Mathematica* (1910), series 3, vol. 10, p. 321.

sentation of imaginaries for which the names of Wessel[1] and Argand[2] preëminently stand, led mathematicians gradually to recognize the *reality* of imaginaries and to feel the need of a more general and a more consequential development of the algebra of imaginaries. The graphic representation of imaginaries had engaged the attention of the Englishman, John Wallis, in the seventeenth century[3] and of the German W. J. G. Karsten in the eighteenth century. Wallis proposed different schemes, none of which satisfactorily visualized vector addition. As previously shown in this history, Karsten concentrated his effort upon the graphic representation of imaginary logarithms, rather than on imaginary numbers in general. In view of these facts the graphic representation due to Wessel and Argand, which was finally adopted by the rank and file of mathematicians through the commanding authority of C. F. Gauss,[4] was a real advance.

The Wessel-Argand diagram does not offer an obvious means of displaying the infinitely many-valued logarithms of a number. Half a century passed before a generalization of this diagram was effected for the display of all the logarithms. For the first half of the nineteenth century there existed no diagram which could take rank with that of the circular and hyperbolic arcs, invented by Karsten in the eighteenth century.

A necessary generalization of the Wessel-Argand diagram was effected by Bernhard Riemann through the surfaces now called "Riemann's surfaces." One purpose of these surfaces was the visualization of the different values of functions of complex variables. The logarithmic function was simply one of several elementary functions to which Riemann's idea applied. Karsten's scheme serves only for the logarithmic function and is therefore less general. Riemann's idea was outlined in his inaugural dissertation, in 1851, at Göttingen, in a paper bearing the title, *Grundlagen für eine allgemeine Theorie der Funktionen einer veränderlichen complexen Grösse*.[5] The reader will picture to himself two planes, one representing the complex numbers z, the other the complex values w, where w is an n-valued function of z. To each point of the z-plane correspond n points of the w plane. It is a purpose of Riemann's scheme to create a geometric representation which avoids the multiplicity of values of a many-valued function and enables one to treat the function as if it were uniform. For this purpose he assumes the w plane to be made up of n sheets or leaves, which together represent the values of w. To each point in one sheet corresponds only one of w; to the n points lying one below the other correspond the n different values of w

[1] *Essai sur la représentation analytique de la direction*, par Caspar Wessel, Copenhague, 1897. Translation of Wessel's memoir in *K. Danske Videnskabernes Selskabs Skrifter, Femte Del. Kjobenhavn*, 1799.

[2] *Essai sur une manière de representer les quantités imaginaires dans les constructions géométriques*, Paris, 1806. An English translation, with historical notes, was brought out by A. S. Hardy, (Van Nostrand) New York, 1881.

[3] G. Eneström, *Bibliotheca mathematica*, 3d S., Vol. 7, 1906–'07, pp. 263–269; *Encyclopédie d. scien. mathématiques*, 1908, T. I., Vol. 1, 5, p. 339, note 51; F. Cajori, *Am. Math. Monthly*, Vol. XIX, 1912, pp. 167–171.

[4] See C. F. Gauss, *Werke*, Bd. II, Göttingen, 1876, p. 95.

[5] B. Riemann, *Math. Werke*, 2. Auflage, Leipzig, 1892, pp. 1–43. See particularly p. 41.

which are gotten from one value of z. If for a certain value of z several values of w become equal to each other, the sheets are supposed to be connected with each other, forming a *branch-point*. For the purpose of exhibiting the continuous passage of one value of w into another, so-called *branch-cuts* are made, which are lines drawn between any two branch-points or from a branch-point to infinity. These branch-cuts are supposed to break the conection, so that there is no passage across them from one side of a sheet to the other side of the same sheet, but they establish a connection between one side of a sheet and the opposite side of some other sheet. Which particular sheets should be connected, depends, of course, upon the properties of the function. These *branch-cuts* and the connections along the cross-cuts between different sheets establish one continuous surface. Riemann makes some brief statements concerning the logarithmic function, but makes no drawing to illustrate the surface to which it gives rise. Applying Riemann's general scheme to the function $w = \log z$, we see that to the z-plane there corresponds a w-surface of infinitely many sheets with all the sheets hanging together at 0 and ∞ and with a branch-cut drawn from 0 to ∞.

Special attention to the graphic representation of the infinitely many values of the general power u^v, where $u = a\ (\cos \alpha + i \sin \alpha)$ and $v = x + iy$, was given by John Warren[1] of Jesus College, Cambridge, and thirty-one years later by H. Durège[2] of the university of Prag. Durège refers to the work of Warren and reëstablishes the latter's result that the points in a plane representing the values of one and the same power all lie upon a logarithmic spiral and are so distributed that the radius vectors of two successive points include the constant angle $2\pi x$. Durège proceeds to a more elaborate study of the wonderful spirals and their vectors.

Somewhat similar studies which culminated in a new graphic representation of the infinitely many logarithms of a complex number—a representation that has found wide acceptance in texts on the theory of functions of complex variables and is often preferred to the visualization by Riemann's surfaces—was given in 1871 by G. Holzmüller of Magdeburg in a paper, *Ueber die logarithmische Abbildung und die aus ihr entspringenden orthogonalen Curvensysteme.* Holzmüller[3] takes $Z = \log z$ and shows that, to each infinitely long strip of area of the Z-plane, drawn parallel to the x-axis, and of width 2π, there corresponds the entire z-plane; that to each strip drawn parallel to the y-axis there corresponds in the Z-plane a circular ring of definite width, which must, however, be taken as wound about itself an infinite number of times. To each point of the Z-plane corresponds only one point of the z-plane; to each point of the z-plane corresponds an infinite number of points of the Z-plane, all of which lie on a line parallel to the y-axis and appear at intervals of 2π. Thus Holzmüller pictures the geometric signi-

[1] J. Warren published two articles in the *Philosophical Transactions* for the year 1829, London, pp. 241 and 339.

[2] H. Durège, in *Zeitsch. f. Math. u. Physik*, Leipzig, Vol. 5, 1860, p. 345.

[3] *Zeitschr. f. Math. u. Phys.*, Leipzig, Vol. 16, 1871, pp. 269–289.

cance of the many-valuedness of the logarithm and the periodicity of the exponential function. He acknowledges suggestions received from a lecture by Schwarz of Zürich. Holzmüller connects these results with the Riemann surfaces, and proceeds to elaborate some of the wonderful properties of the logarithmic spiral and to show the connection of this with the Ptolemaic stereographic projection and loxodromic lines.

The use of the logarithmic spiral in the graphic representation of complex numbers and their logarithms to any base, real or complex, upon the Wessel-Argand plane, is explained by Irving Stringham and M. W. Haskell in papers which we discuss more fully under the classification of logarithmic systems.

[The next instalment will treat the general power and logarithm.]

CERTAIN THEOREMS IN THE THEORY OF QUADRATIC RESIDUES.

By D. N. LEHMER, University of California.

The definition of a quadratic residue is usually given as follows: *If an integer X can be found to satisfy the congruence,*

$$X^2 \equiv D (\mathrm{mod}\ m)$$

where D is relatively prime to m, then D is said to be a quadratic residue of m. The restriction that D shall be relatively prime to m simplifies many results. There are theories, however, in which this restriction serves to cloud the results and in this paper we will not impose it. We proceed, first of all, to determine the number of residues for a given integer m, using the enlarged definition.

We take first of all the case where the modulus m is a power of an odd prime p. In the first place there will arise $\frac{1}{2}\varphi(p^\alpha)$ distinct residues from the squaring of the numbers k, which are less than p^α, and prime to p, $p^\alpha - k$ and k giving the same residues. Consider next the numbers kp, where k is prime to p. If two such numbers when squared give the same residue, we have:

$$k^2p^2 \equiv {k_1}^2p^2 (\mathrm{mod}\ p^\alpha),$$

whence,

$${k_1}^2 \equiv {k_1}^2 (\mathrm{mod}\ p^{\alpha-2})$$

or

$$k \equiv \pm k_1 (\mathrm{mod}\ p^{\alpha-2}).$$

Give now k the $\varphi(p^{\alpha-2})$ values less than $p^{\alpha-2}$ and prime to p. The resulting squares furnish $\frac{1}{2}\varphi(p^{\alpha-2})$ distinct residues. In the same way the numbers kp^2 where k is prime to p furnish $\frac{1}{2}\varphi(p^{\alpha-4})$ distinct residues which also are different from those obtained from kp obtained above. Proceeding in this way we obtain for the total number of distinct residues:

$$\tfrac{1}{2}[\varphi(p^\alpha) + \varphi(p^{\alpha-2}) + \varphi(p^{\alpha-4}) + \varphi(p^{\alpha-6}) + \cdots] + 1$$

the unit being added for the residue zero.